JOURNAL OF APPROXIMATION THEORY **88**, 304–315 (1997)
ARTICLE NO. AT963023

Interpolation by Convex Algebraic Hypersurfaces

Eugenii Shustin

School of Mathematical Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel

Communicated by Nira Dyn

Received December 28, 1994; accepted in revised form February 27, 1996

Let Σ be the set of vertices of a convex non-degenerate polyhedron in $\mathbf{R}^n$, $n \geqslant 2$. We suggest an algorithm to construct smooth convex algebraic hypersurfaces of degree as small as possible, going through Σ.

INTRODUCTION

The standard algebraic interpolation problem in $\mathbf{R}^n$ is to find an algebraic hypersurface of degree as small as possible, passing through a given finite set Σ. The convex algebraic interpolation problem is: Given a finite set $\Sigma \subset \mathbf{R}^n$ in convex position, we look for a smooth convex algebraic hypersurface of degree as small as possible, going through Σ. A Hermite-type convex algebraic interpolation problem is to find a smooth convex algebraic hypersurface, going through Σ and tangent at Σ to given hyperplanes. A modification of this problem—interpolation by convex piecewise algebraic curves and surfaces—was treated in [2–6, 11]. In [8] a solution to the convex algebraic interpolation problem in $\mathbf{R}^2$ is presented: namely, a family of convex curves of degree $[(m+1)/2]$, going through all vertices of a convex m-gon, is constructed. This algorithm gives a solution to the Hermite-type convex algebraic interpolation problem in $\mathbf{R}^2$ as well, but in general does not work for higher dimensions.

In the present paper we suggest an approach to both ordinary and Hermite-type convex algebraic interpolation problems in any dimension $n \geqslant 2$ using hyperbolic hypersurfaces. The advantage is that, given a real polynomial of degree >2, it is hard to check whether the polynomial defines a convex hypersurface or a hypersurface with a convex connected component (see Section 2 below) and the hyperbolicity is a well-controllable property. An explicit equation of convex hypersurfaces that we construct below can be found from some systems of non-linear equations which have a simple linear approximation.

0021-9045/97 $25.00

I am very grateful to Professor D. Levin for introducing to the subject and inspiring this work and to Professor N. Alon for helpful discussions. I thank the referee for his useful remarks and suggestions.

Throughout the article we number statements and equations separately.

1. FORMULATION OF RESULTS

To simplify the notation in the following we denote an algebraic hypersurface and a polynomial, defining it by the same symbol.

A finite set $\Sigma \subset \mathbf{R}^n$ is called *convexly located* if no point of Σ belongs to the convex hull of the other points. The convex hull $\mathscr{C}(\Sigma)$ of Σ is a convex polyhedron.

A connected component of a smooth algebraic hypersurface in $\mathbf{R}^n$ is called *convex* if all its finite subsets are convexly located. An algebraic hypersurface H is said to be *convex interpolatory* for a convex set Σ if H has a smooth convex connected component containing Σ. If such a component of H is bounded then the hypersurface H will be called *bounded convex interpolatory* for Σ.

Let $\Sigma \subset \mathbf{R}^n$ be a convexly located set of m points. We denote by $d(\Sigma)$ the minimal number of disjoint proper faces of $\mathscr{C}(\Sigma)$ such that they all are simplices (not necessary equidimensional) and their union contains Σ. Such a set of faces we will call a covering set. It is easy to see that for $n=2$

$$d(\Sigma) \leqslant \frac{m+1}{2}. \tag{1}$$

If $n=3$ then from [1]

$$d(\Sigma) < \tfrac{2}{3}m, \tag{2}$$

and the constant 2/3 is tight. For any $n \geqslant 4$,

$$d(\Sigma) < m, \tag{3}$$

and this estimate is asymptitically tight; i.e., there are examples with

$$d(\Sigma) = m - O(m^{1/[n/2]}).$$

These examples, shown to me by Professor N. Alon, are presented in the proof of Theorem 5 below.

Our main result is the following

THEOREM 1. *Let $\Sigma \subset \mathbf{R}^n$ be a set of $m > n$ convexly located points, and* $\dim \mathscr{C}(\Sigma) = n$. *Then there exists a hypersurface in $\mathbf{R}^n$ of degree $d(\Sigma)$, convex* *interpolatory for* Σ, *and there exists a hypersurface in* $\mathbf{R}^n$ *of degree* $\leqslant d(\Sigma)+1$, *bounded convex interpolatory for* Σ.

THEOREM 2. *Let* $\Sigma = \{z_1, ..., z_m\} \subset \mathbf{R}^n$, $m \geqslant n$, *be a convexly located set, and let* $L_1, ..., L_m$ *be a set of hyperplanes in* $\mathbf{R}^n$ *such that*

$$L_i \cap \mathscr{C}(\Sigma) = \{z_i\}, \qquad i = 1, ..., m.$$

There exist a hypersurface $F \subset \mathbf{R}^n$ *of degree* m, *convex interpolatory for* Σ, *and a hypersurface* $G \subset \mathbf{R}^n$ *of degree* $\leqslant m+1$, *bounded convex interpolatory for* Σ, *such that* F *and* G *are tangent to* L_i *at* z_i *for all* $i = 1, ..., m$.

The same approach allows us to solve a problem of the mixed type as well. For a proper nonempty subset $\Sigma' \subset \Sigma$ let us denote by $d(\Sigma', \Sigma)$ the minimal number of proper faces of $\mathscr{C}(\Sigma)$ such that they all are simplices and their union contains $\Sigma \backslash \Sigma'$.

THEOREM 3. *Let* $\Sigma' = \{z_1, ..., z_p\}$ *be a subset of a convexly located set* $\Sigma \subset \mathbf{R}^n$ *of* $m \geqslant n$ *points. Let* $L_1, ..., L_p$ *be a set of hyperplanes in* $\mathbf{R}^n$ *such that*

$$L_i \cap \mathscr{C}(\Sigma) = \{z_i\}, \qquad i = 1, ..., p.$$

There exist a hypersurface $F \subset \mathbf{R}^n$ *of degree* $p + d(\Sigma', \Sigma)$, *convex interpolatory for* Σ, *and a hypersurface* $G \subset \mathbf{R}^n$ *of degree* $\leqslant p + d(\Sigma', \Sigma) + 1$, *bounded convex interpolatory for* Σ, *such that* F *and* G *are tangent to* L_i *at* z_i *for all* $i = 1, .., p$.

In the case $n = 2$ Theorem 1 gives a convex interpolatory curve for Σ of degree $[(m+1)/2]$, whereas, actually, in [8] a bounded convex interpolatory curve of the same degree was constructed. The following statement specifies this result.

THEOREM 4. *For any convexly located set* $\Sigma \subset \mathbf{R}^2$ *of* $m \geqslant 4$ *points there exist a convex interpolatory curve of degree* $[m/2]$ *and a bounded convex interpolatory curve of degree* $[(m+1)/2]$.

The following statement together with estimates (1), (2), (3) shows how optimal are the results of Theorems 1 and 4 with respect to the degree of interpolatory hypersurfaces.

THEOREM 5. (1) *For any* $m \geqslant 4$ *there exists a convexly located set* $\Sigma_m \subset \mathbf{R}^2$ *of* m *points such that there is no convex smooth curve through* Σ_m *of degree* $< [m/2]$.

(2) *For any* $m \geqslant 10$ *there exists a convexly located set* $\Sigma_m \subset \mathbf{R}^3$ *of* m *points such that there is no convex smooth surface through* Σ_m *of degree* $< [(2m-4)/3]$.

(3) *For any* $n \geqslant 4$ *and* $m \geqslant n+1$ *there exists a convexly located set* $\Sigma_m \subset \mathbf{R}^n$ *of* m *points such that there is no convex smooth hypersurface through* Σ_m *of degree less than* $m - s_0$, *where*

$$s_0 = \min \left\{ s \geqslant n+1 \mid s + \binom{s - [(n+1)/2]}{s-n} + \binom{s - [n/2] - 1}{s-n} \geqslant m \right\}.$$

Remarks. (1) In the case of odd m there may not exist a bounded convex interpolatory curve of degree $(m-1)/2$. For example, five points on a hyperbola determine this hyperbola as the unique conic curve through the chosen points.

(2) In the third statement of Theorem 5

$$m - s_0 \geqslant \binom{s_0 - 1 - [(n+1)/2]}{s_0 - 1 - n} + \binom{s_0 - 2 - [n/2]}{s_0 - 1 - n} - 1,$$

where the right-hand side is a polynomial in s_0 of degree $[n/2]$. Hence the statements of Theorems 1 and 5 mean that there are convexly located sets $\Sigma \subset \mathbf{R}^n$, $n \geqslant 4$, of m points with $d(\Sigma) = m - O(m^{1/[n/2]})$.

(3) The convex interpolatory curves and hypersurfaces, which we construct in the proofs of Theorems 1–4, are not unique. For instance, by Theorem 4, given a convexly located set $\Sigma \subset \mathbf{R}^2$ of $m = 2k+1$ points, there exists a convex interpolatory curve C_k of degree k through Σ. For $k \geqslant 3$ the space of curves of degree k (being the space of polynomials in two variables of degree k, taken up to a constant factor) has dimension $k(k+3)/2 > 2k+1$; hence the set of curves of degree k, going through Σ, is a projective space of a positive dimension. and hereby any such curve, sufficiently close to C_k, is a convex interpolatory for Σ as well.

2. HYPERBOLIC POLYNOMIALS

Let q be a point in the real projective space $\mathbf{R}P^n$. A real homogeneous polynomial $F(x_0, ..., x_n)$ of degree d is called *q-hyperbolic* (*strict q-hyperbolic*) if the hypersurface $F = 0$ in $\mathbf{R}P^n$ intersects any straight line through q at d real points counting multiplicities (resp., at d distinct real points).

LEMMA 1. *Any q-hyperbolic polynomial is the limit of strict q-hyperbolic polynomials of the same degree.*

Proof. In fact, this statement is due to Nuij [9]. We will present here the required family of strict q-hyperbolic polynomials in a slightly modified form, suggested in [10]. Let $F(x_1, ..., x_n)$ be a q-hyperbolic polynomial of degree d, and let $q = (1, 0, ..., 0)$. Then according to [9] the polynomials

$$\mathcal{T}_\xi F(x_0, ..., x_n) = \left(\mathrm{Id} + \xi x_1 \frac{\partial}{\partial x_1}\right)^d \circ \cdots \circ \left(\mathrm{Id} + \xi x_n \frac{\partial}{\partial x_n}\right)^d F(x_0, ..., x_n), \quad (4)$$

where Id is the identity operator, are strict q-hyperbolic for all constants $\xi \in \mathbf{R} \setminus \{0\}$, while $\mathcal{T}_0 F = F$. ∎

LEMMA 2. *Let F be a strict q-hyperbolic polynomial of degree d if $d = 2k$ then the hypersurface F in $\mathbf{R}P^n$ consists of k smooth connected components homeomorphic to the $(n-1)$-sphere S^{n-1}. Each connected component bounds in $\mathbf{R}P^n$ a domain homeomorphic to the n-dimensional ball D^n and containing q, and all these balls form an ascending sequence. If $d = 2k+1$ then the hypersurface F consists of $k+1$ smooth connected components, k of them are homeomorphic to S^{n-1} and situated as described above; one more component is homeomorphic to $\mathbf{R}P^{n-1}$ and does not bound any part of $\mathbf{R}P^n$.*

Proof. The fact is well known in real algebraic geometry. We will explain it shortly. First, it is known (see, for example [13]) that a smooth real hypersurface consists of orientable components homologous to zero in $\mathbf{R}P^n$, and, in the case of odd degree, contains one more component realizing a non-zero homology class. At last, note that the natural projection of a strict q-hyperbolic hypersurface onto the space $\mathbf{R}P^{n-1}$ of lines going through q is a d-sheeted covering, which completes the proof because $\mathbf{R}P^{n-1}$ can be covered either by $\mathbf{R}P^{n-1}$ or by S^{n-1}, and, for $n > 2$, a component $\mathbf{R}P^{n-1}$ cannot bound anything in $\mathbf{R}P^n$, because it intersects any (projective) straight line L through q at one point and hence does not divide L into two or more connected components. ∎

In particular, a strict q-hyperbolic hypersurface of degree >1 has a unique component homeomorphic to S^{n-1} which bounds a component of $\mathbf{R}P^n \setminus F$ homeomorphic to a ball. We will call it the *inner component*.

LEMMA 3. *Given a strict q-hyperbolic hypersurface F, let S be its inner component. For any hyperplane $H \subset \mathbf{R}P^n$, the set $S \setminus H \subset \mathbf{R}^n = \mathbf{R}P^n \setminus H$ consists either of one or two convex components of the affine hypersurface $F \setminus H$.*

Proof. Note that F is q'-hyperbolic with respect to any point q' belonging to the ball bounded by the inner component. Therefore any straight line in $\mathbf{R}P^n$ meets the inner component at at most two points, and we are done. ∎

3. PROOF OF THEOREM 1

3.1. *Construction of a Convex Interpolatory Hypersurface*

Let $\sigma_1, ..., \sigma_d$, $d = d(\Sigma)$, be a covering set of faces of $\mathscr{C}(\Sigma)$. Through these faces one can draw hyperplanes $H_1, ..., H_d$ such that

$$H_i \cap \mathscr{C}(\Sigma) = \sigma_i, \qquad i = 1, ..., d.$$

Let $\tilde{H}_1, ..., \tilde{H}_d \subset \mathbf{R}P^n$ be the projective closures of $H_1, ..., H_d$, respectively.

It is clear that the hypersurface

$$F = \tilde{H}_1 \cdots \tilde{H}_d$$

is q-hyperbolic with respect to any point q in the interior of $\mathscr{C}(\Sigma)$. In order to get a strict q-hyperbolic polynomial we apply the deformation (4). The problem is how to keep these hyperbolic hypersurfaces from passing through Σ. So we will slightly modify the deformation (4).

Let $\Sigma = \{z_1, ..., z_m\}$. In a neighborhood of each point z_i, $i = 1, ..., m$, on the straight line $\langle qz_i \rangle$ we introduce a local coordinate s_i such that the coordinate of z_i is 0.

LEMMA 4. *There exist $\varepsilon > 0$ and smooth families $\tilde{H}_{1,\bar{s}}, ..., \tilde{H}_{d,\bar{s}}$ of hyperplanes, depending on parameters*

$$\bar{s} = (s_1, ..., s_m), \qquad |s_i| < \varepsilon, \qquad i = 1, ..., m,$$

such that $\tilde{H}_{j,0} = \tilde{H}_j$, $j = 1, ..., d$, and, for any pair $z_i \in \tilde{H}_j$, the hyperplane $\tilde{H}_{j,\bar{s}}$ meets the line $\langle qz_i \rangle$ at the point with coordinate s_i.

Proof. This follows immediately from the fact that the points of Σ belonging to $\tilde{H}_j$ are verices of a simplex, hence are linearly independent. ∎

Let us consider the family of hypersurfaces $\mathscr{T}_\xi F_{\bar{s}}$, where the operator $\mathscr{T}_\xi$ is defined by (4), q is assumed to be (1. 0, ..., 0), and

$$F_{\bar{s}} = \tilde{H}_{1,\bar{s}} \cdots \tilde{H}_{d,\bar{s}},$$

$$\bar{s} = (s_1, ..., s_m), \qquad |\xi| < \delta, \quad |s_1|, ..., |s_m| < \varepsilon,$$

with some fixed positive δ, ε. These are strict q-hyperbolic hypersurfaces of degree d for all $\xi \neq 0$. Now we seek $s_1, ...,. s_m$ as functions of ξ such that $s_i(0) = 0$, $i = 1, ..., m$, and, for any $\xi \in (-\varepsilon, \varepsilon)$, the hypersurface $\mathscr{T}_\xi F_{\bar{s}}$ contains Σ.

Homogeneous polynomials in $n+1$ variables of degree d, close to F_0, can be parametrized by the collection of their coefficients

$$\mathscr{A} = \{A_{i_0, ..., i_n}, i_0 + \cdots + i_n = d, i_0 < d\},$$

assuming $A_{d,0,\dots,0} = \text{const.} \neq 0$. By construction, the coefficients $\mathscr{A}$ of the polynomials $\mathscr{T}_\xi F_{\bar{s}}$ are smooth functions of $\xi, \bar{s}$ in a neighborhood of zero. On the other hand, since any straight line $\langle qz_i \rangle$ meets F_0 transversally at distinct points (by choice of q in a generic position), this straight line meets transversaly each hypersurface $\mathscr{T}_\xi F_{\bar{s}}$ at d distinct points. Therefore, the coordinate S_i of the intersection point of the line $\langle qz_i \rangle$ and $\mathscr{T}_\xi F_{\bar{s}}$, which is close to z_i, is a smooth function of $\mathscr{A}$. Thereby, our problem can be reformulated as to find a solution $s_1(\xi), \dots, s_m(\xi)$ of the system

$$S_i(\mathscr{A}(\xi, s_1, \dots, s_m)) = 0, \qquad i = 1, \dots, m,$$

in a neighborhood of zero. Note that

$$S_i(\mathscr{A}(0, s_1, \dots, s_m)) = s_i, \qquad i = 1, \dots, m,$$

for all $s_1, \dots, s_m$ close to zero. Hence

$$\det \left(\frac{\partial S_i}{\partial s_j} \right)_{1 \leqslant i, j \leqslant m} \Bigg|_{\xi = 0, \bar{s} = 0} = 1,$$

therefore by the implicit function theorem there exists a solution $s_1(\xi), \dots, s_m(\xi)$, defined on some interval $\xi \in (-\delta, \delta)$ and satisfying

$$s_i(0) = 0, \quad |s_i(\xi)| < \varepsilon, \qquad \xi \in (-\delta, \delta), \quad i = 1, \dots, m.$$

That completes the construction of a convex interpolatory hypersurface.

3.2. *Construction of a Bounded Convex Interpolatory Hypersurface*

If the hyperplanes $H_1, \dots, H_d$, introduced above, bound a compact polyhedron which contains Σ, then the previous procedure gives a bounded inner component of the hyperbolic hypersurface constructed.

Assume that the component of the complement to $H_1 \cup \cdots \cup H_d$ in $\mathbf{R}^n$, containing q, is unbounded. Since the points of Σ lying in H_1 are vertices of a simplex, there exists an $(n-2)$-sphere S^{n-1} in H_1 going through these points. Obviously, there exists an $(n-1)$-sphere S^{n-1} in $\mathbf{R}^n$ going through S^{n-2} and bounding a ball which contains Σ. Thus, substituting H_1 for S^{n-1} and performing the procedure from Section 3.1, we get a bounded convex interpolatory hypersurface of degree $d(\Sigma) + 1$.

4. PROOF OF THEOREMS 2 AND 3

We will perform, actually, the above procedure. Let $\tilde{H}_1, \dots, \tilde{H}_m \subset \mathbf{R}P^n$ be the projective closures of the affine hyperplanes $L_1, \dots, L_m$. Define the normal vector of a hyperplane $a_0x_0+a_1x_1+\cdots+a_nx_n=0$ in $\mathbf{R}P^n$ with $a_0\neq 0$ as

$$v=\left(\frac{a_1}{a_0},\ ...,\ \frac{a_n}{a_0}\right)\in\partial\mathbf{R}^n.$$

Denote by $\bar{v}_1, ..., \bar{v}_m$ the normal vectors of $\tilde{H}_1, ..., \tilde{H}_m$, and introduce families

$$\tilde{H}_1(s_1, \bar{w}_1), ..., \tilde{H}_m(s_m, \bar{w}_m)$$

of hyperplanes depending on parameters $s_1, ..., s_m \in \mathbf{R}$ close to zero, and vectors

$$\bar{w}_i=(w_{i1}, ..., w_{in})\in\mathbf{R}^n, \qquad i=1, ..., m,$$

close to $\bar{v}_1, ..., \bar{v}_m$, respectively, such that the hyperplane $\tilde{H}_i(s_i, \bar{w}_i)$ meets the line $\langle qz_i\rangle$ at the point with coordinate s_i and has the normal vector $\bar{w}_i$, $i=1, ..., m$.

As in Section 3.1, we look for the required convex interpolatory hypersuface F in the family $\mathscr{T}_\xi F(\bar{s}, \bar{w}_1, ..., \bar{w}_m)$, where

$$F(\bar{s}, \bar{w}_1, ..., \bar{w}_m)=\prod_{i=1}^{m}\tilde{H}_i(s_i, \bar{w}_i), \qquad \bar{s}=(s_1, ..., s_m).$$

Clearly, the coefficients of F

$$\mathscr{A}=\{A_{i_0, ..., i_n},\ i_0+\cdots+i_n=m,\ i_0\neq m\}$$

are smooth functions of $\bar{s}, \bar{w}_1, ..., \bar{w}_m$, if $A_{m,0,...,0}=\text{const}\neq 0$. On the other hand, the coordinates $S_1, ..., S_m$ of the intersection points of F with the lines $\langle qz_1\rangle, ..., \langle qz_m\rangle$ in neighborhoods of points $z_1, ..., z_m$, respectively, and the normal vectors $V_1, ..., V_m$ of the tangent hyperplanes to F at these intersection points depend smoothly on $\mathscr{A}$. Thus, our problem is reduced to solution of the system

$$S_i(\mathscr{A}(\xi, \bar{s}, \bar{w}_1, ..., \bar{w}_m))=0, \qquad V_i(\mathscr{A}(\xi, \bar{s}_1, \bar{w}_1, ..., \bar{w}_m))=\bar{v}_i, \qquad i=1, ..., m, \tag{5}$$

with respect to $\bar{s}, \bar{w}_1, ..., \bar{w}_m$ as functions of ξ. Since

$$S_i(\mathscr{A}(0, \bar{s}, \bar{w}_1, ..., \bar{w}_m))=s_i,$$

$$V_i(\mathscr{A}(0, \bar{s}, \bar{w}_1, ..., \bar{w}_m))=\bar{w}_i, \qquad i=1, ..., m,$$

the Jacobian of the left-hand sides of (5) with respect to $s_1, \ldots, s_m$, $w_{11}, \ldots, w_{mn}$ is non-degenerate which, by the implicit function theorem, provides the existence of the required solution to (5) which completes the construction of F.

If the hyperplanes $L_1, \ldots, L_m$ bound a compact polyhedron in $\mathbf{R}^n$ containing Σ, then we put $G = F$. Otherwise, we substitute the hyperplanek L_1 for a $(n-1)$-sphere, tangent to L_1 at z_1 and embracing Σ, in the construction described above, and get a bounded convex interpolatory hypersurface G of degree $m+1$, which completes the proof of Theorem 2.

The proof of Theorem 3 is a simple combination of the proofs of Theorems 1 and 2.

5. PROOF OF THEOREM 4

5.1. *Existence of a Bounded Convex Interpolatory Curve*

In [8] a convex interpolatory curve of degree $[(m+1)/2]$ was constructed. We will show that this algorithm gives also a bounded convex interpolatory curve. Assume that $m = 2k$ $(k \geqslant 2)$. Let us number successively the edges of the m-gon $\mathscr{C}(\Sigma)$ and put

$$F_\lambda = \lambda \prod_{i=1}^{k} H_{2i} + (1-\lambda) \prod_{i=1}^{k} H_{2i-1}, \qquad \lambda = \text{const} \in (0, 1),$$

where H_i is the straight line through the ith edge, $i = 1, \ldots, m$. The fact that all the curves F_λ, $\lambda \in (0, 1)$, are convex interpolatory for Σ is proved in [8, Theorem 1]. It was shown in the proof of Theorem 1 in [8] that the convex component of F_λ lies in the closure of the set $(\Pi_1 \cup \Pi_2) \backslash (\Pi_1 \cap \Pi_2)$, where

$$\Pi_1 = \bigcap_{i=1}^{k} \pi_{2i-1}, \qquad \Pi_2 = \bigcap_{i=1}^{k} \pi_{2i},$$

and $\pi_i \subset \mathbf{R}^2$, $1 \leqslant i \leqslant m$, denotes the closed half plane, bounded by H_i and containing $\mathscr{C}(\Sigma)$. It is not difficult to see that there exists a straight line $H \subset \mathbf{R}^2$, which bounds a half plane π such that $\pi \supset \mathscr{C}(\Sigma)$, and $\pi \cap (\Pi_1 \cup \Pi_2)$ is bounded. Now we shift H in a parallel way, keeping the above property, until $H \cap \mathscr{C}(\Sigma) = \{z_i\}$, $z_i \in \Sigma$. Note that the tangent to F_λ at $z_i = H_i \cap H_{i+1}$ runs over that interval (H_i, H_{i+1}) of the line pencil through z_i, which contains H, as λ varies in the interval $(0, 1)$. Choosing $\lambda \in (0, 1)$ so that F_λ is tangent to H at z_i, we get a convex curve through Σ lying in the bounded set $(\Pi_1 \cup \Pi_2) \cap \pi$.

The case of odd m can be considered analogously.

5.2. *Existence of a Convex Interpolatory Curve*

We only have to construct a convex interpolatory curve of degree k for a set Σ, consisting of $m = 2k+1$ $(k \geqslant 2)$ points. We will use a slightly modified procedure from [8]. Let $\Sigma = \{z_1, ..., z_{2k+1}\} \subset \mathbf{R}^2$ be the set of successively numbered vertices of a convex $(2k+1)$-gon. Introduce the straight lines

$$H_1 = \langle z_1, z_2 \rangle, \qquad H_2 = \langle z_2, z_3 \rangle, ..., H_{2k-1} = \langle z_{2K-1}, z_{2k} \rangle,$$
$$H_{2k} = \langle z_{2k}, z_1 \rangle.$$

As in the previous subsection, we define the family F_λ of curves of degree k and the sets Π_1, Π_2, assuming that π_i, $i = 1, ..., 2k$, is the closed half plane, bounded by H_i and containing the points $z_1, ..., z_{2k}$. Clearly, the point z_{2k+1} belongs to $(\Pi_1 \cup \Pi_2) \backslash (\Pi_1 \cap \Pi_2)$. Since the curves F_λ cover the interior of the latter set as λ runs through the interval (0, 1), there exists $\mu \in (0, 1)$ such that the curve F_μ (convex by [8, Theorem 1]) does through z_{2k+1}, which completes the construction.

6. PROOF OF THEOREM 5

(1) Let us consider a convexly located set $\Sigma_m \subset \mathbf{R}^2$, consisting of $m-1$ points on a convex conic curve C and of one more point outside the disk bounded by C. Then any curve F of degree

$$d \leqslant \left[\frac{m}{2}\right] - 1 = \left[\frac{m-2}{2}\right],$$

going through Σ_m, meets C at least at

$$m - 1 > 2 \cdot \frac{m-2}{2} \geqslant 2d$$

points, hence, by Bezout's theorem [12], F must contain C as component and cannot be interpolatory for Σ.

(2) Let $m = 3s_0 - 1 - r \geqslant 10$, where $r = 3$, 4, or 5, and s_0 is an integer. Clearly, there exists a convex polyhedron Δ in $\mathbf{R}^3$ with s_0 vertices z_i, $i = 1, ..., s_0$, such that one of its facets (faces of codimension 1) is an r-angle and the other are triangles. Denote by s_1, s_2 the numbers of edges and facets of Δ, respectively. From

$$3s_2 + r - 3 = 2s_1, \qquad s_0 - s_1 + s_2 = 2,$$

one derives $s_2 = 2s_0 - r - 1$. Denote by w_i, $i = 1, ..., s_2$, the baricenters of the facets of Δ, and by $\bar{v}_i$, $i = 1, ..., s_2$, the normal vectors of the corresponding facets, oriented in the exterior of Δ. For a given $\varepsilon > 0$ denote by $w_i(\varepsilon)$ the point $w_i + \varepsilon \bar{v}_i$, 1, ..., s_2. For a sufficiently small $\varepsilon_1 < 0$ the sets

$$\Sigma_m(\varepsilon) = \{z_1, ..., z_{s_0}, w_1(\varepsilon), ..., w_{s_2}(\varepsilon)\}, \qquad 0 < \varepsilon < \varepsilon_0,$$

of m points are convexly located in $\mathbf{R}^2$. Put $d(\varepsilon)$ to be the minimal degree a convex interpolary surface through $\Sigma_m(\varepsilon)$. This integral-valued function defines a semi-algebraic subdivision of the interval $(0, \varepsilon_0)$. Hence there is $\varepsilon_1 \in (0, \varepsilon_0)$ such that

$$d(\varepsilon) = d^* = \text{const}, \qquad \varepsilon \in (0, \varepsilon_1).$$

Since the (projective) space of real surfaces of degree d is compact, there exists a sequence $H_1, H_2, H_3, ...$ of convex interpolatory surfaces of degree d^* for the sets $\Sigma_m(\varepsilon_1), \Sigma_m(\varepsilon_2), \Sigma_m(\varepsilon_3), ...$, respectively, such that $\lim \varepsilon_k = 0$ and there exists $\lim H_k = H \neq 0$. Since $w_i(\varepsilon_k) \to w_i$ as $k \to \infty$, the convexity condition implies that the limit shape of the convex component H_k must be Δ, hence H contains the planes through all the s_2 facets of Δ as components. Thereby,

$$d^* \geqslant s_2 = 2s_0 - r - 1 = \left\lceil \frac{2m-4}{3} \right\rceil,$$

which completes the proof of the second part.

(3) Let us fix $n \geqslant 4$. The convex hull $\Delta(s)$ of $s > n$ generic points $z_1, ..., z_s$ on the curve

$$\{(t, t^2, ..., t^n) \in \mathbf{R}^n \mid t \in \mathbf{R}\}$$

is a so-called cyclic polyhedron with q vertices and

$$\mu(s, n) = \binom{s - [(n+1)/2]}{s-n} + \binom{s - [n/2] - 1}{s-n}$$

facets [7]. For a given $m \geqslant n+1$, let

$$s_0 = \min\{s \geqslant n \mid s + \mu(s, n) \geqslant m\}.$$

Clearly, $m \geqslant s_0 \geqslant n+1$, since $\mu(n, n) = 2$. Put $r = m - s_0$. As above, we introduce the normal vectors $\bar{v}_i$, $i = 1, ..., r$, of r distinct facets $\sigma_1, ..., \sigma_r$ of

$\Delta(s_0)$, oriented in the exterior of $\Delta(s_0)$. Also, we fix one point w_i inside each facet σ_i, $i=1, ..., r$. For a sufficiently small $\varepsilon>0$ the set $\Sigma_m(\varepsilon)$ of m points

$$z_i, \quad i=1, ..., s_0, \qquad w_i(\varepsilon)=w_i+\varepsilon\bar{v}_i, \quad i=1, ..., r,$$

is convexly located in $\mathbf{R}^n$. As above one shows that the minimal degree of a convex interpolatory hypersurface for $\Sigma_m(\varepsilon)$ is a constant d^* as $\varepsilon \in (0, \varepsilon_1)$, and $d^* \geqslant r$, which implies the required statement.

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